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Application of methods of conditional multidimensional minimization to the ballistic trajectory calculation problem

The paper focuses on the problem of calculating the approximate ballistic missile trajectory, the calculation ensuring that the missile travels from a given launch point to the finish point and covering the entire range rate for the missiles of the type considered. The missile trajectory is defined by a system of nonlinear differential equations. A different range is achieved by changing the initial values of the flight-path angle and the operating time of the missile stages. Due to the physical significance, these variables are constrained. The problem of multidimensional conditional minimization by the method of barrier functions with minimization of Nelder – Meed method.

Keywords: ballistic missile trajectory, conditional multidimensional minimization.

Introduction

The paper focuses on the problem of calculating the approximate trajectory of a ballistic missile from a given launch point to the finish point, covering the entire range rate for the missile type in question.

An intercontinental ballistic missile (ICBM) with a range rate of 5000 to 10,000–13,000 km (depending on the model) was selected as an example. Each range corresponds to certain values of launch parameters (angle of attack, operation time of missile stages, flight-path angle). Usually, a missile is controlled by the angle of attack. The parameter’s time dependence is described by a function depending on the missile model. There is no public data on this function.

The trajectory calculation can be simplified to a great extent if we assume the angle of attack is equal to zero and if we select variable parameters, which can be set discretely (flight-path angle and operation time of each missile stage). It is assumed that by changing the parameters listed above, we can reach the specified range rate with a selected accuracy, while maintaining compliance with physical realizability requirements and reference data.

System of equations for ICBM trajectory calculation

A surface-to-surface intercontinental ballistic missile has been selected as the research object.

The following assumptions have been made:

- the Earth is a spherical body;
- the Earth’s rotation is taken into account;
- the atmospheric model is exponential;
- the drag force is constant for each missile stage;
- gravity acceleration remains unchanged regardless of latitude and includes only the radial component, but it changes depending on altitude;
- aerodynamic coefficients are constant.

With account for the specified assumptions in projections on the axes of half-speed coordinate system, the equations of centre of mass movement composed relative to the observed velocity are represented as follows [1, 2]:

$$\frac{dm}{dt} = \mu;$$

$$m\frac{dv}{dt} = P\cos\alpha - \frac{\rho v^2}{2} S\left(C_x^{\alpha^2}\alpha^2 + C_x\right) - \\ - mg\sin\theta - mr(\cos\theta\cos\psi_a\cos\varphi + \\ + \sin\theta\sin\varphi); \qquad (1)$$

$$mv\frac{d\theta}{dt} = P\sin\alpha + \frac{\rho v^2}{2} SC_y^{\alpha}\alpha - mg\cos\theta - \\ - 2mv\omega\sin\psi_a\cos\varphi - m\omega^2 r\sin\varphi \times \\ \times(\cos\theta\sin\varphi - \sin\theta\cos\psi_a\cos\varphi) + \frac{v^2}{r} m\cos\theta;$$

$$\rho = \rho_0 e^{\frac{-(r-R_z)}{7170}};$$

$$\frac{dr}{dt} = \nu \sin\theta;$$

$$\frac{d\lambda}{dt} = \frac{-\nu\cos\theta\sin\psi_a}{r\cos\varphi};$$

$$\frac{d\varphi}{dt} = \frac{\nu\cos\theta\cos\psi_a}{r}; \quad (1)$$

$$g = G\frac{Mm}{r^2}.$$

Here, m – missile weight including fuel (kg);

t – time (s);

μ – fuel consumption rate (kg/s);

ν – missile velocity (m/s);

P – drag force being constant until fuel is depleted, then it is equal to zero (kg·m/s^2);

α – angle of attack, i.e. the angle between the velocity vector ν projection on the symmetry plane and longitudinal axis of aircraft (deg);

ρ – atmosphere density, $\rho_0 = 1.225$ (kg/m^3);

S – reference area (m^2);

$C_x^{\alpha^2}$, C_x, C_y^{α} – aerodynamic coefficients (1/deg^2, non-dimensional, 1/deg, respectively);

g – gravity (kg·m/s^2);

θ – flight-path angle (deg);

r – distance to the Earth's centre (m);

ψ_a – launch azimuth, i.e. heading from launch point to end point (rad);

φ – latitude (deg);

ω – Earth's angular velocity equal to $7.292115078 \cdot 10^{-5}$ (s^{-1});

R_z – Earth's radius equal to 6,378,245 (m);

λ – latitude (deg);

G – gravity constant equal to $6.67408(31)\cdot 10^{-11}$ (m^{3}s^{-2}kg^{-1});

M – Earth's mass equal to $5.97219 \cdot 10^{24}$ (kg).

The system of equations (1) is solved by numerical integration. As the calculation accuracy criterion, a residual given as the function of the initial flight-path angle and operation time of each missile stage was selected:

$$J(\theta_{нач}, t_1, t_2, t_3) = \left((\varphi_{кон}) - \varphi(\theta_{нач}, t_1, t_2, t_3)\right)^2 + \\ + \left((\lambda_{кон}) - \lambda(\theta_{нач}, t_1, t_2, t_3)\right)^2,$$

where $\theta_{нач}$ – initial value of flight-path angle;

t_1, t_2, t_3 – operation time of the first, second and third missile stages, respectively;

$\varphi_{кон}$, $\lambda_{кон}$ – preset coordinates of the end point;

$\varphi(\theta_{нач}, t_1, t_2, t_3)$, $\lambda(\theta_{нач}, t_1, t_2, t_3)$ – end point coordinates determined by solving the system of equations.

This criterion defines the accuracy needed to bring an aircraft to the selected space point and is represented as a function of several variables.

Due to physical reasons, these variables are constrained as follows:

$$45° \le \theta_{нач} \le 89°,\ 0 \le t_1 \le t_{1\max},$$

$$0 \le t_2 \le t_{2\max},\ 0 \le t_3 \le t_{3\max},$$

where $t_{1\max}$, $t_{2\max}$, $t_{3\max}$ – maximum operation time of the first, second and third missile stages, respectively.

Therefore, in order to solve the problem, methods of conditional multidimensional minimisation shall be applied.

Methods of conditional multidimensional minimisation

The target function is given as a system of differential equations, therefore, it is impossible to apply a series of methods which involve the target function derivative. The variable replacement method is also inadequate for the problem in question. Moreover, we considered methods based on reducing the minimisation problem with constraints to the problem of minimisation of a function without constraints. An auxiliary function is introduced as the sum of the function to be minimised and the penalty function with account for constraints.

Method of penalty functions

An auxiliary function is to be selected, matching a given function to be minimised within the admissible domain and rapidly increasing beyond it:

$$F(\mathbf{x}, l) = f(\mathbf{x}, l) + \sum\nolimits_{i=1}^{n} \psi_i \left(g_i(\mathbf{x}), l \right),$$

where $f(\mathbf{x}, l)$ – initial function to be minimised;

$\mathbf{x} = [x_0, \ldots, x_n]$;

n – number of variables;

l – a vector parameter, $l = \{l_i\}$, $i = \overline{1, n}$;

$g_i(\mathbf{x})$ – constraints, $g_i(\mathbf{x}) \le 0$.

Here, $\psi_i \left(g_i(\mathbf{x}), l_i \right)$ is the penalty function with certain properties [3]:

$$\lim_{l_i \to \infty} \left(\psi_i \left(g_i(\mathbf{x}), l_i \right) \right) = \\ = \begin{cases} 0 & \text{при } g_i(\mathbf{x}) \le 0, \ i = \overline{1, n}; \\ +\infty & \text{в противном случае.} \end{cases}$$

This method requires additional studies to determine the functions such as $\psi_i \left(g_i(\mathbf{x}), l_i \right)$ and values l_i, $i = \overline{1, n}$.

Method of barrier functions

This method is represented as follows

$$F(\mathbf{x}, l) = f(\mathbf{x}, l) - k \sum_{i=1}^{n} \frac{1}{g_i(\mathbf{x})}, \quad k > 0.$$

When $\mathbf{x}$ approaches the boundaries of domain X (from the inside), the values of at least one of the bound functions approach zero from the domain of negative values. In this case, a large positive value is added to function $f(\mathbf{x})$. At $k \to 0$, the minimum of function $F(\mathbf{x}, k)$ tends to the minimum of function $f(\mathbf{x})$ with constraints $g_i(\mathbf{x}) \le 0$ [3]. A significant advantage of the method is that its application for calculating an auxiliary function does not require additional studies. That is why the method of barrier functions was selected.

To determine the minimum of the resulted auxiliary function, the Nelder – Meed method was applied.

Results of ICBM trajectory calculations

The ICBM trajectory was calculated with the help of numerical integration, using the two-step Euler’s method with the second-order accuracy (integration step of 0.0005 s), the method of barrier functions and minimisation of the difference between calculated and preset coordinates of the finish point using the Nelder – Meed method. The Minuteman ICBM data was used as source data [4, 5]. The resulted trajectories for various ranges are shown in the figure.

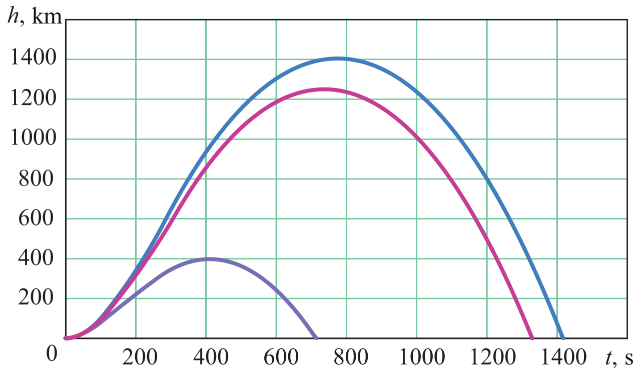

Figure. Results of ICBM trajectory calculation for various ranges (km):
– 5000; – 9000; – 12,000

Conclusion

The obtained results comply with physical significance requirements and reference data. Therefore, we may conclude that the method of barrier functions along with Nelder – Meed method allows to cover the required ballistic missile flight range rate with the accuracy corresponding to the ballistic missile performance characteristics.

The problem has been solved using the “Fort” software product and is used for recognising various operational and tactical situations in order to gather timely and accurate information for aerospace warning. The calculated ballistic missile trajectories are displayed on 2D and 3D maps.

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Применение методов условной многомерной минимизации к задаче расчета траектории баллистической ракеты

Рассмотрена задача расчета приблизительной траектории баллистической ракеты, обеспечивающего попадание ракеты из заданной точки старта в точку финиша и охватывающего весь диапазон дальностей для ракет рассматриваемого типа. Траектория ракеты задана системой нелинейных дифференциальных уравнений. Достижение различной дальности обеспечено изменением начальных значений угла наклона траектории и времени работы ступеней. В связи с физическим смыслом на эти переменные наложены ограничения. Решена задача многомерной условной минимизации методом барьерных функций с минимизацией методом Нелдера – Мида.

Ключевые слова: траектория баллистической ракеты, условная многомерная минимизация.

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