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Mathematical model of tracking systems taking into account nonlinearities in state variables

Mathematical modelling makes it possible to verify correct operation of the system being designed before actually implementing it, but linear models do not enable us to evaluate the effect of structural constraints, taking which into account makes irregularities appear. We present an approach that allows us to design and implement a model of a dynamic tracking system using the coefficients of the desired transfer function to account for irregularities in the momentum and state variables. We present results of modelling in the *MATLAB/Simulink* environment taking various irregularities into account.

Keywords: tracking system, desired transfer function, state-space representation, nonlinearities of state variables, Cauchy normal form.

Introduction

When designing tracking systems, it is important to ensure that the system being designed operates correctly. Modelling is an effective tool for evaluating the parameters of a tracking system, selecting technical solutions and setting requirements for the components of tracking systems. It is not difficult to synthesize a controller in the class of linear systems for a controlled object [1] without constraints. One of the most widely used synthesis methods uses the desired transfer function which, as a system function, describes the behaviour of the system as a whole. However, the presence of design constraints for the controlled object [2, 3] places the system in the class of nonlinear [2].

Typically, the following approach is used for modelling the performance of a tracking system with regard to nonlinearities: a controller device is synthesized in a class of linear systems using a given controlled object and calculated desired transfer function, then nonlinearities are established in the controlled object model and quality indicators of tracking systems are evaluated considering the available design constraints [4]. However, in the early stages of design, the structure and most of the parameters of the controlled object are often unknown, so it is difficult to apply the described approach.

The given article presents an approach to create a model considering the design constraints on state variables (velocity and acceleration) and load moment nonlinearity. The proposed approach requires no specific knowledge of the controlled object and uses the coefficients of the desired transfer function of the linear system. This approach is convenient in the early stages of design.

Mathematical model of a tracking system

As mentioned above, the basis for the linear model of a tracking system is the desired transfer function of a closed tracking system which has the form of

$$W_{зам}(p) = \frac{\Delta_{k-1}\omega_0 p^{k-1} \ldots \Delta_1 \omega_0^{k-1} p + \Delta_0 \omega_0^k}{\Delta_1 p^l + \Delta_{l-1}\omega_0 p^{l-1} \ldots \Delta_1 \omega_0^{l-1} p + \Delta_0 \omega_0^l} \quad (1)$$

where Δ_i – standard transfer function coefficients derived according to the table of standard transfer function coefficients;

k, l – the polynomial order of the numerator and denominator of the desired transfer function for a closed system, respectively;

$\omega_0 = tmp/tp$ – the time scaling factor, considering the correction time;

tmp – a table parameter;

tp – correction time.

The transfer function (1) describes the operation of a closed tracking system (Fig. 1, a). The desired transfer function can be calculated using different methods with various criteria of the quality of tracking system functioning. For instance, one can specify quality indicators (correction time

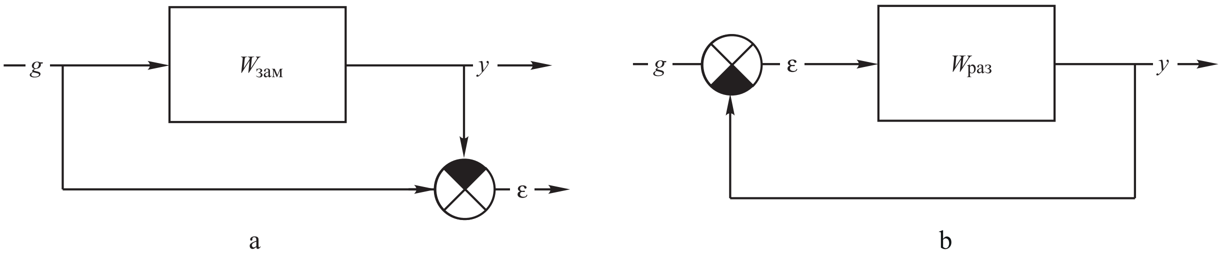

Fig. 1. Block diagram of the closed (a) and open (b) tracking system:
ε – error signal; g – input action; y – system response

and overcorrection) and requirements for tracking with zero errors of polynomial influences of various degrees for astatic systems.

In practice, it is often necessary to model the systems with structures corresponding to open tracking systems such as the one shown in Fig. 1, b.

The transfer function of such a tracking system has the form

$$W_{раз}(p) = \frac{W_{зам}}{1 - W_{зам}} = \frac{\eta_m p^m + \eta_{m-1} p^{m-1} + \ldots + \eta_1 p + \eta_0}{\delta_n p^n + \delta_{n-1} p^{n-1} + \ldots + \delta_1 p + \delta_0}, \quad (2)$$

where η, δ – the numerator and denominator coefficients of the open system transfer function, respectively; their values are determined by the coefficients of the desired transfer function (1);

m, n – polynomial order of the numerator and denominator of the open system transfer function, respectively.

For the purposes of obtaining the ability of imposing constraints and considering nonlinearities in the state variables and load moment, it is necessary to find a representation of the system in the form of differential equations system in the Cauchy normal form [5]. Let us proceed from the open system transfer function (2) to the differential equation

$$y^{(n)} + \frac{\delta_{n-1}}{\delta_n} y^{(n-1)} + \ldots + \frac{\delta_0}{\delta_n} y = \frac{\eta_m}{\delta_n} g^{(m)} + \frac{\eta_{m-1}}{\delta_n} g^{(m-1)} + \ldots + \frac{\eta_0}{\delta_n} g. \quad (3)$$

The differential equation (3) corresponds to a system of differential equations in the Cauchy normal form

$$\begin{cases} \dot{y}_1 = y_2 + F_1 g; \\ \dot{y}_2 = y_3 + F_2 g; \\ \ldots \\ \dot{y}_{n-1} = y_n + F_{n-1} g; \\ \dot{y}_n = -a_0 y_1 - a_1 y_2 - \ldots - a_{n-1} y_n + F_n g \end{cases} \quad (4)$$

or in the matrix form

$$\begin{cases} \dot{y} = \boldsymbol{A} y + \boldsymbol{B} g; \\ y = \boldsymbol{C} y. \end{cases}$$

In system (4) $y_1 = y$, $y_2 = \frac{dy}{dt}$, ..., $y_n = \frac{d^{n-1} y}{dt^{n-1}}$; $a_0 = \frac{\delta_0}{\delta_n}$, ..., $a_{n-1} = \frac{\delta_{n-1}}{\delta_n}$, $a_n = 1$. Functions $F_i(t)$, $i = 1, \ldots, n$ are calculated using recurrence formula

$$F_i(t) = b_{n-1}(t) - \sum_{k=0}^{i-1} \sum_{s=0}^{i-k} C_{n+s-i}^{n-i} a_{n-i+k+s}(t) \frac{d^s F_k}{dt^s},$$

where

$$C_{n+s-i}^{n-i} = \frac{(n-i+s)!}{(n-i)!\,s!},$$

and F_k are calculated as follows:

$$F_0 = b_n,$$

$$F_1 = b_{n-1} - a_{n-1} F_0,$$

$$F_2 = b_{n-2} - a_{n-1}F_1 - a_{n-2}F_0,$$

$$\vdots$$

$$F_j = b_{n-j} - \sum_{m=0}^{j-1} a_{n-j+m}F_m,$$

where $b_0 = \frac{n_0}{\delta_n}, \ldots, b_m = \frac{\eta_m}{\delta_n}$.

With the given choice of state variables, the system matrices have the form

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \ldots & -a_{n-1} \end{pmatrix},$$

$$\boldsymbol{B}^T = \left(F_1(t) \quad F_2(t) \quad \ldots \quad F_{n-1}(t) \quad F_n(t)\right)^T,$$

$$\boldsymbol{C}^T = \left(1 \quad 0 \quad \ldots \quad 0\right).$$

Fig. 2 shows a block diagram plotted based on the system of differential equations in Cauchy normal form (4).

Let us now consider the mathematical description of typical nonlinearities that are supposed to be used in modelling of design constraints (state variables and drive moment nonlinearities).

The saturation type nonlinearity model with respect to the state variable $y^{(k)}$ has the form

$$\bar{y}^{(k)} = \begin{cases} y^{(k)}, & \left|y^{(k)}\right| < y^{(k)}_{\max}; \\ y^{(k)}_{\max}\mathrm{sign}\,(y^{(k)}), & \left|y^{(k)}\right| \geq y^{(k)}_{\max}. \end{cases}$$

Let us introduce a constraint model for the minimum state variable $y^{(k)}$

$$\bar{y}^{(k)} = \begin{cases} 0, & \left|y^{(k)}\right| \leq y^{(k)}_{\min}; \\ y^{(k)}, & \left|y^{(k)}\right| \geq y^{(k)}_{\min} \end{cases}$$

and a moment nonlinearity model

$$M_{\text{с}} = M_{\text{н}0}(ky + 1), \qquad (5)$$

where $M_{\text{н}0}$ – rated (average) value of the load moment;

k – coefficient of load moment dependence on output variable (e.g., on the drive shaft rotation angle).

Let us also consider the dry friction model of the actuator shaft. The static moment of resistance for the mechanism is represented as a nonlinear function of four variables [6, 7]:

$$M_{\text{с}} = M_{\text{с}}(\omega, M_{\text{тр}}, M_{\text{д}}), \qquad (6)$$

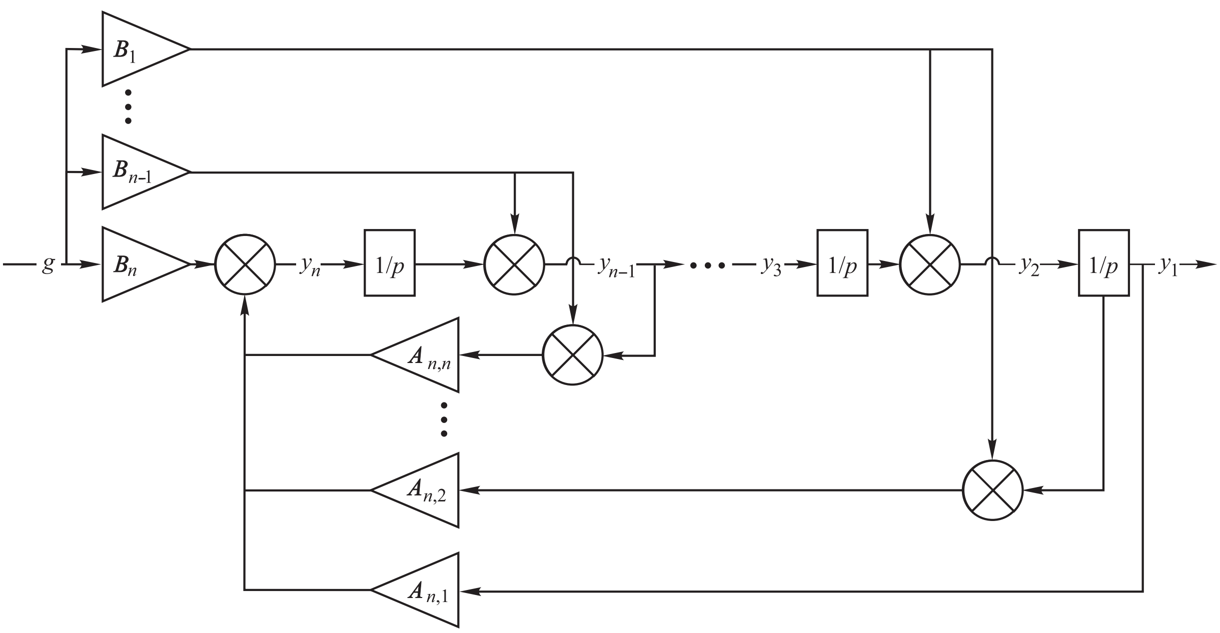

Fig. 2. Block diagram of differential equation system in Cauchy normal form; coefficients $A_{n,i}$ and B_j are elements of matrix $\boldsymbol{A}$ and vector column $\boldsymbol{B}$

where $M_{тр}$ – dry sliding friction moment (constant positive value);

$M_{д}$ – motor moment.

Function (6) is represented as

$$M_{с} = \begin{cases} M_{тр}\operatorname{sign}\omega & \text{при } \omega \neq 0; \\ f(M_{д}) & \text{при } \omega = 0; \end{cases}$$

$$f(M_{д}) = \begin{cases} M_{д} & \text{при } |M_{д}| \leq M_{тр}; \\ M_{тр}\operatorname{sign}(M_{д}) & \text{при } |M_{д}| > M_{тр}. \end{cases}$$

Such moment representation allows considering nonlinear properties of friction forces and inelastic deformation both at motion and at rest, including the conditions for starting and stopping of mechanisms.

Software model of a tracking system

Based on the algorithm described above, a model of the tracking system has been designed and implemented in *MATLAB/Simulink* environment, considering the velocity and acceleration constraints of the system as well as nonlinearities of the drive load moment. The coefficients of the differential equation, constraint parameters, the design constant of the electrical drive and the moment of inertia can all be established as parameters. The *Simulink* model is shown in Fig. 3.

As an example, let us consider an astatic tracking system of $l = 3$, order, possessing the $v_g = 2$, order of astatism based on the input action, which is described by the desired transfer function according to formula (1) and the numerical values of coefficients [8] $\Delta_0 = 1$, $\Delta_1 = 6.35$, $\Delta_2 = 5.1$, $\Delta_3 = 1$ and $\omega_0 = 1$

$$W_{зам}(p, \omega_0) = \\ = \frac{\Delta_1\omega_0^2 p + \Delta_0\omega_0^3}{\Delta_3 p^3 + \Delta_2\omega_0 p^2 + \Delta_1\omega_0^2 p + \Delta_0\omega_0^3} = \\ = \frac{6.35p + 1}{p^3 + 5.1p^2 + 6.35p + 1}, \quad (7)$$

The problems set forth are to evaluate the quality performance of an astatic tracking system with respect to nonlinearities and degradation of these performance indicators compared with the linear one.

The design constraints and parameters of the system are considered:

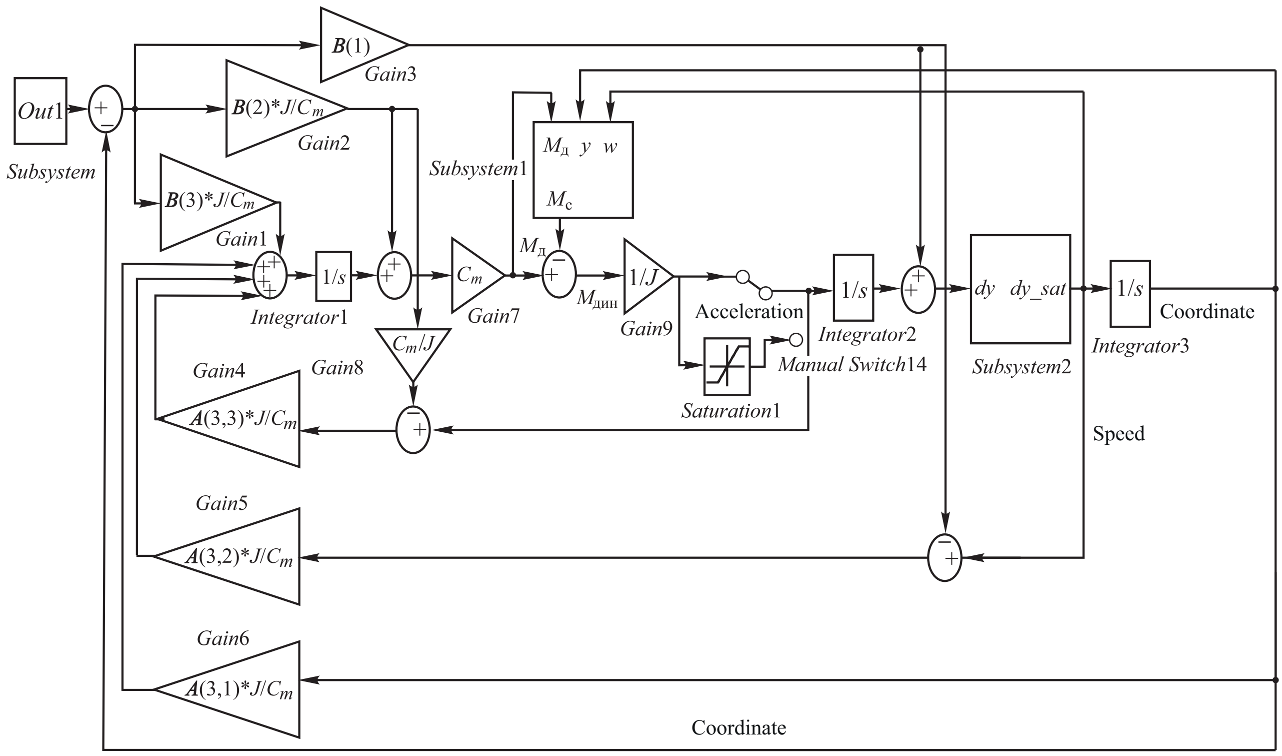

Fig. 3. *Simulink* model diagram:
$M_{дин}$ – dynamic moment

• constraint on the maximum acceleration and velocity value $\ddot{y}_{max} = 2$ rad/s^2; $\dot{y}_{max} = 0.6$ rad/s;

• constraint on the minimum velocity value $|\dot{y}_{min}| = 0.05$ rad/sec;

• moment nonlinearity of the form (5), where $M_{н0} = 0.1$ N·m, $k = 0.2$;

• moment nonlinearity considering dry friction according to nonlinear function (6);

• design constant of electrical drive $C_m = 1.82$ N·m/A and moment of inertia, respectively, $J = 0.022$ N·m^2.

Considering (2) and (7), let us proceed to the open system transfer function

$$W_{раз}(p) = \frac{6.35p + 1}{p^3 + 5.1p^2}. \qquad (8)$$

The transfer function (8) corresponds to the differential equation

$$6.35\dot{g} + g = \dddot{y} + 5.1\ddot{y}. \qquad (9)$$

Then, according to the algorithm described above, the matrices of the differential equations system in Cauchy normal form, corresponding to differential equation (9), have the form

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5.1 \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 6.35 \\ -31.385 \end{pmatrix},$$

and the system is written as follows:

$$\begin{cases} \dot{y}_1 = y_2; \\ \dot{y}_2 = y_3 + 6.35g; \\ \dot{y}_3 = -0y_1 - 0y_2 - 5.1y_n - 31.385g. \end{cases} \qquad (10)$$

Using system (10), we developed a model in *MATLAB/Simulink* environment (see Fig. 3), where $A(3.1) = 0$, $A(3.2) = 0$, $A(3.1) = -5.1$, $B(1) = 0$, $B(2) = 6.35$, $B(3) = -31.385$ according to (10).

The following blocks are used in the diagram:

• input action *Subsystem* (Fig. 4, a);

• velocity nonlinearity *Subsystem2* (Fig. 4, b);

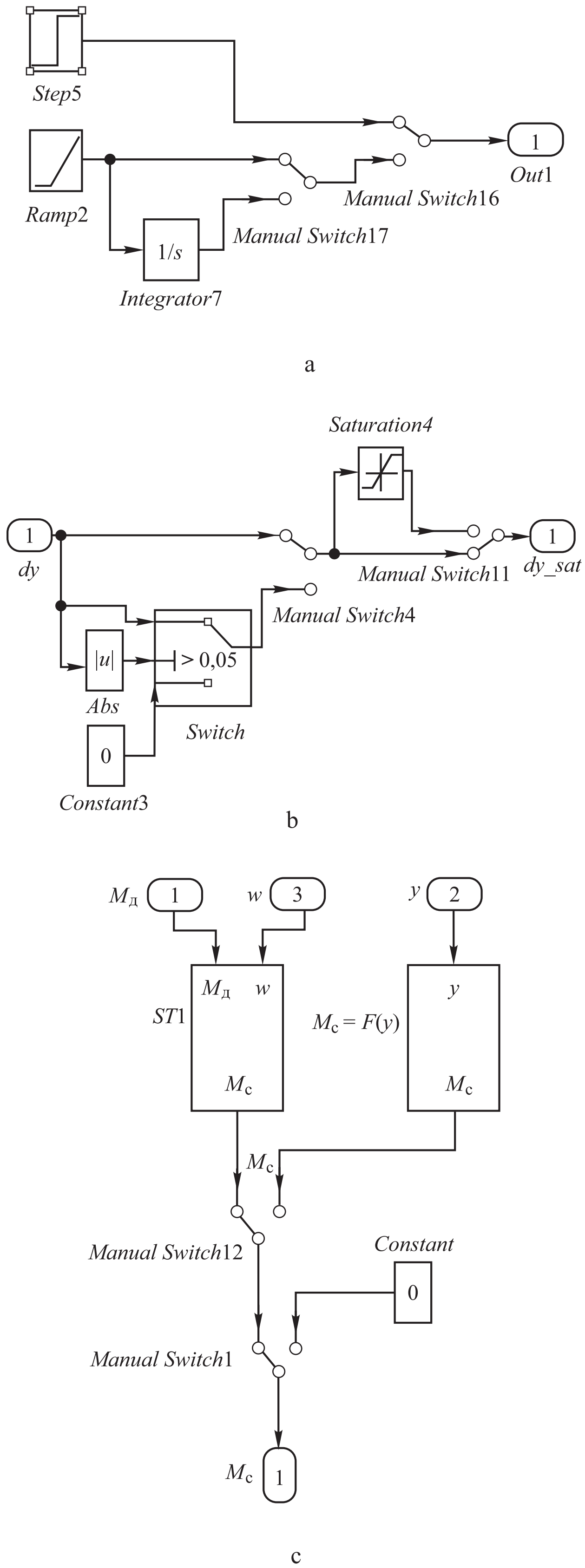

Fig. 4. *Subsystem* (a), *Subsystem2* (b) and *Subsystem1* (c): *ST1* – nonlinearity according to nonlinear function (6); $M_c = F(y)$ – nonlinearity by moment according to equation (5)

- nonlinear moment *Subsystem*1 (Fig. 4, c);
- *Integrator*1, *Integrator*2, *Integrator*3;
- *Gain*1, *Gain*2, *Gain*3, *Gain*4, *Gain*5, *Gain*6, *Gain*7, *Gain*8, *Gain*9;
- saturation type element *Saturation*1.

Analysis of the results

Several experiments were carried out with different sets of nonlinearities using the implemented model.

1. Nonlinearity of the saturation type in terms of acceleration and velocity, i.e. constraint

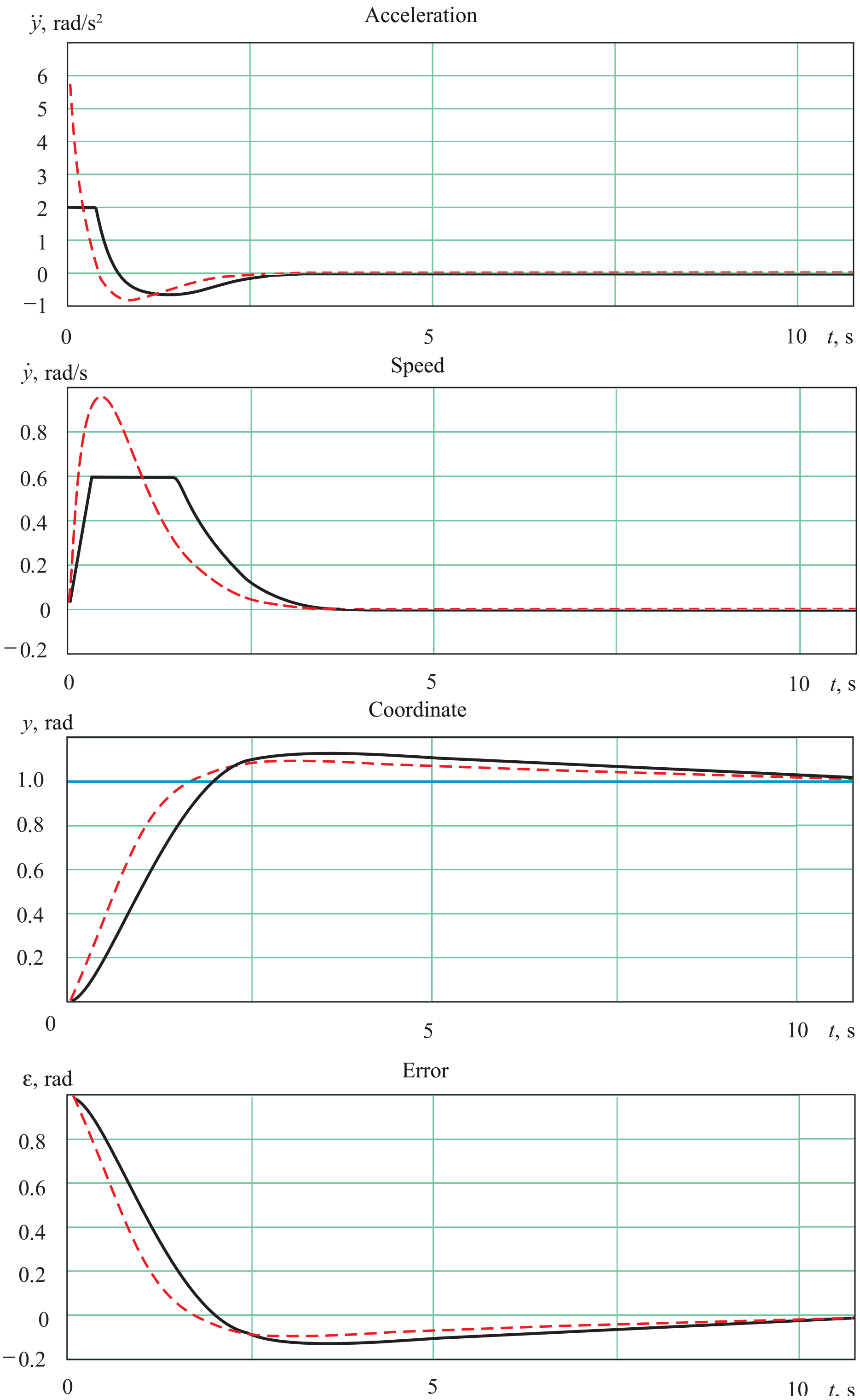

Fig. 5. System behaviour with velocity and acceleration nonlinearities of the "saturation" type: – –, —— – behaviour of linear and nonlinear systems; —— – input action

on the maximum value of the state variable (Fig. 5). In the acceleration and velocity graphs, straight sections can be seen that are not present in the linear system. These are the sections where the constraints have been implemented. Nonlinearities of the saturation type in terms of velocity and acceleration resulted in a prolongation of the transient process and an increase in overcorrection relative to the linear system.

2. Nonlinearity of the dead band type, i.e. constraint on the minimum velocity value (Fig. 6). Nonlinearity of this type causes the system to oscillate in acceleration and velocity when the system operates at low speeds, resulting in a staircase output signal. Relative to the linear system, the transient process in a nonlinear system will also be delayed.

3. Nonlinearity in terms of load moment of the form $M_c = F(y)$. The behaviour of the system is shown in Fig. 7, a. The dependence of load moment on the angle is given in Fig. 7, b. In the linear system, the load moment remained constant. As can be seen, an increase in load moment with increasing coordinate causes a delay

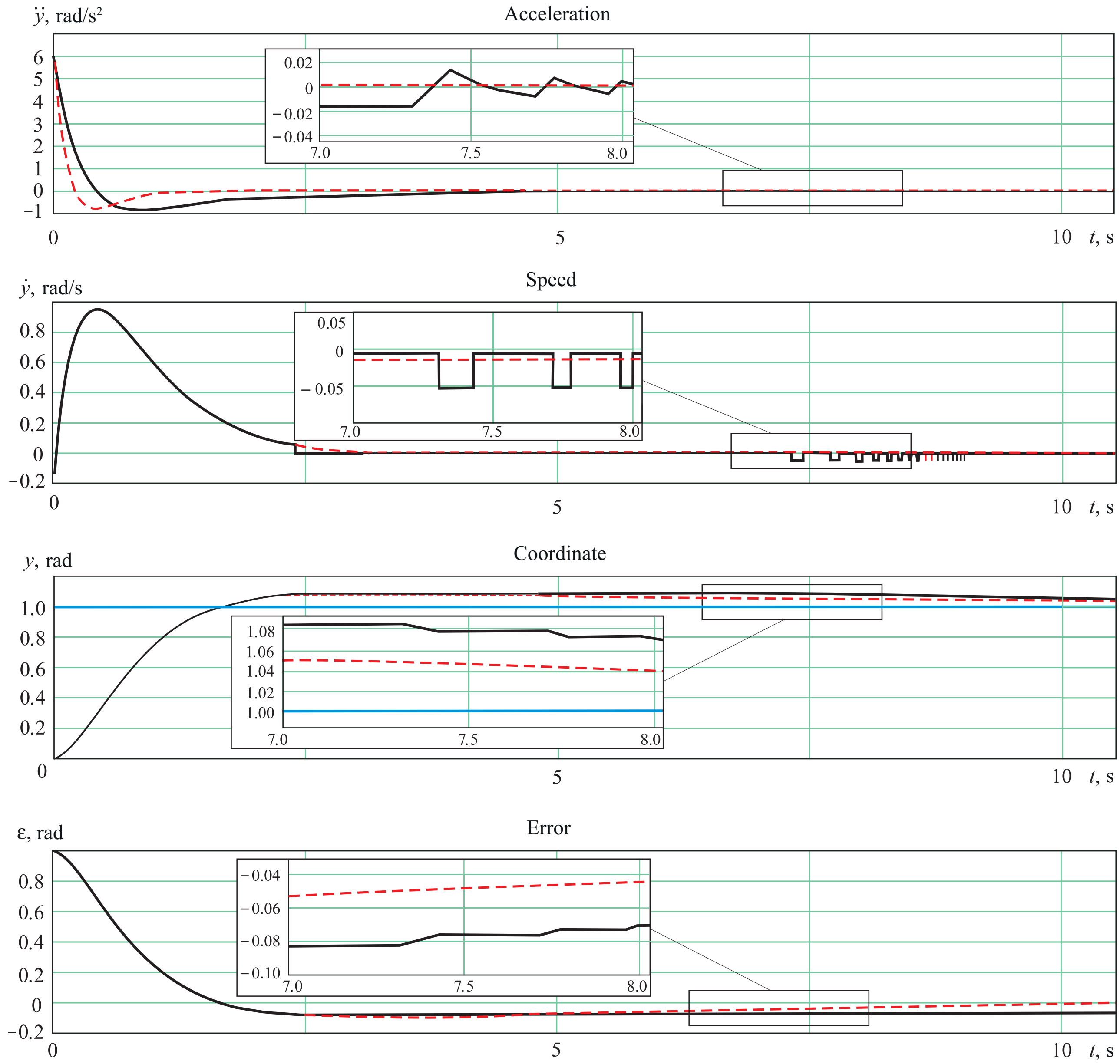

Fig. 6. Behaviour of the system with a minimum velocity constraint:
– –, —— – behaviour of linear and nonlinear systems;
—— – input action

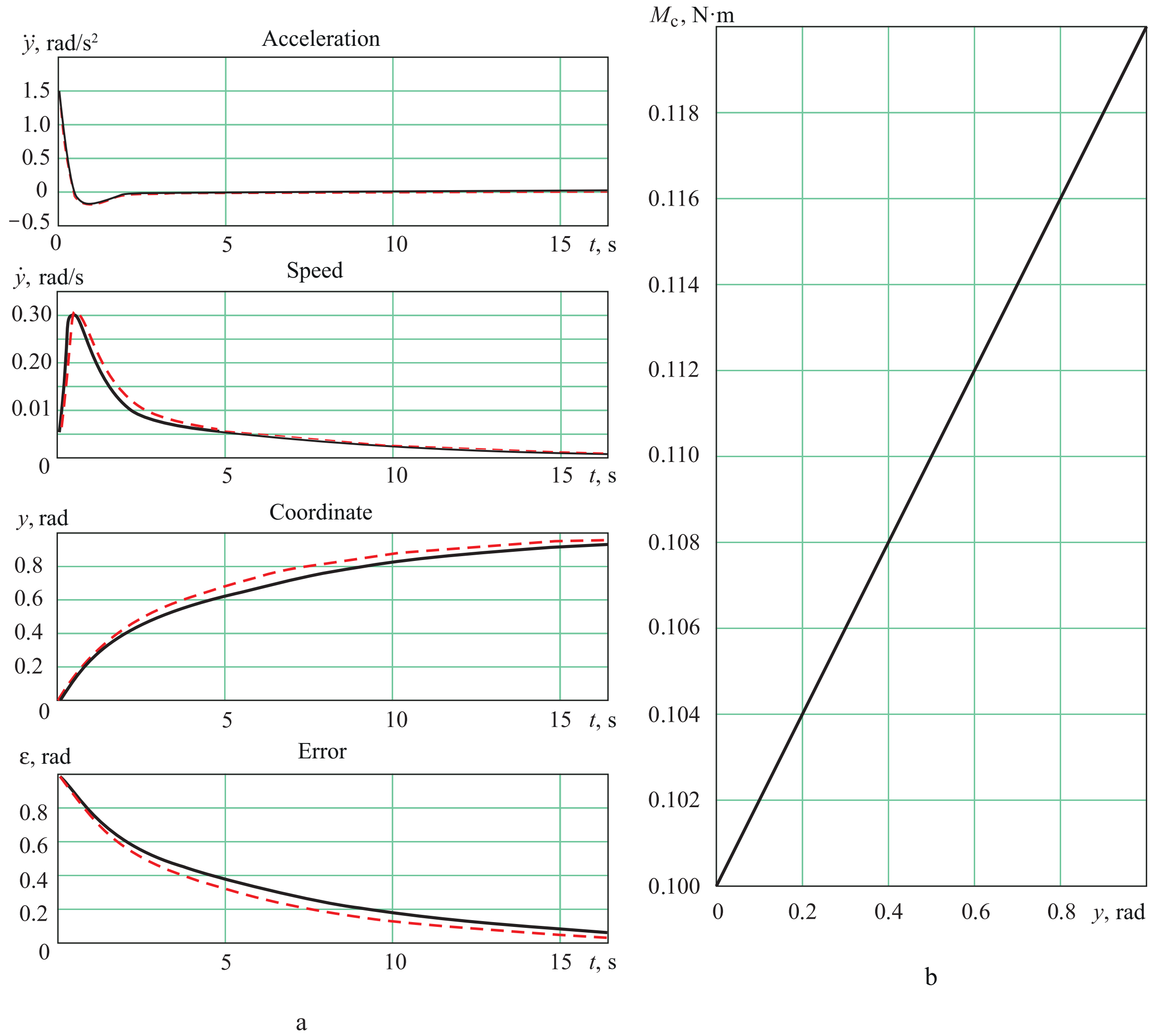

Fig. 7. Behaviour of the system with nonlinearity of the $M_н = F(y)$ form (a) and the dependence of the moment $M_c = F(y)$ on the angle coordinate (b)

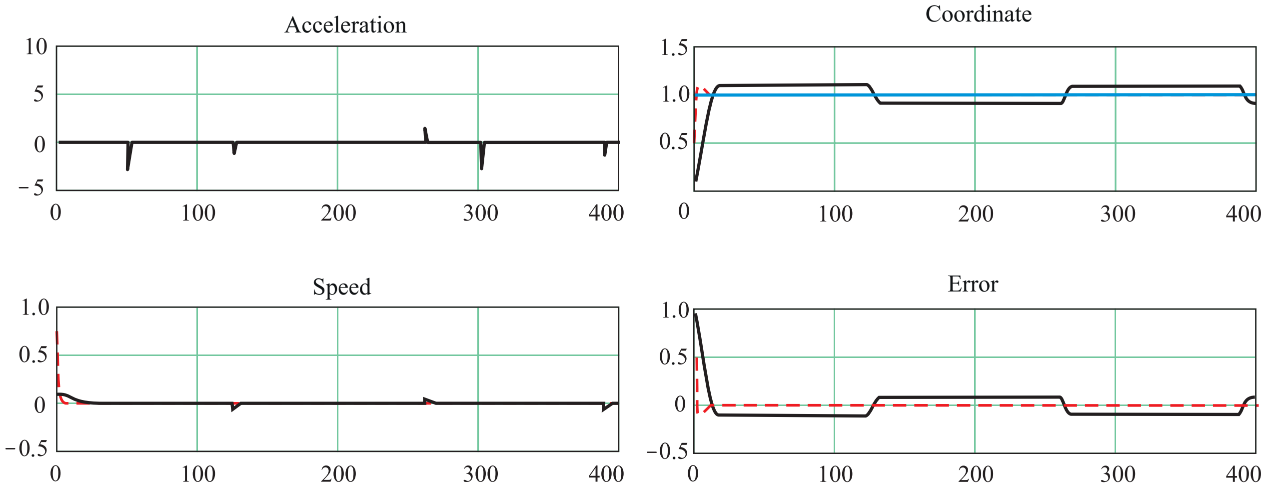

Fig. 8. Behaviour of the system with nonlinear moment considering friction and elastic deformation forces: – –, —— – behaviour of linear and nonlinear systems; —— – input action

in the transient process with respect to the linear system at the constant load moment.

4. Nonlinearity in terms of moment with regard to friction forces. The behaviour of the system is shown in Fig. 8. The transient process has increased significantly compared to the linear system. Additional to that, there are self-oscillations both in terms of acceleration and velocity as well as in terms of the coordinate when the system operates at low speeds.

Conclusions

The mathematical model of a dynamic tracking system considering the nonlinear state variables has been developed. In contrast to other known tracking systems, it is based on a linear closed tracking system model in the form of a system function, namely, the desired transfer function.

The desired transfer function can be developed based on various criteria, e.g. for astatic systems with established correction time and overcorrection. The model can be used to simulate the operation of tracking systems with different nonlinear state variables, system engineering calculations at the early stages of design, requirements to the components of tracking systems (e.g. actuator parameters, minimum velocity of the system, non-uniformity of the mechanical part, etc.).

A software parametric model is implemented in *MATLAB/Simulink* environment with the ability to investigate the behaviour of both linear state variables and those having the following nonlinearities:

• saturation in terms of acceleration and velocity (i.e., constraint on the maximum value);

• velocity dead band (i.e., constraint on the minimum value);

• nonlinearity in terms of load moment of $M_н = F(y)$ type (load moment versus dependence on the drive shaft angle of rotation);

• nonlinearity in terms of moment, considering friction forces.

Mathematical modelling has been performed to demonstrate the application of the described approach to the practical task of assessing the quality of tracking system operation.

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Математическая модель следящих систем с учетом нелинейностей переменных состояния

С помощью математического моделирования можно проверить правильность работы проектируемой следящей системы перед технической реализацией, однако линейные модели не позволяют оценить влияние конструктивных ограничений, учет которых приводит к появлению нелинейностей. Представлен подход, позволяющий спроектировать и реализовать модель динамической следящей системы с учетом нелинейностей по моменту и переменных состояния, при котором используются коэффициенты желаемой передаточной функции. Приведены результаты моделирования в среде *MATLAB/Simulink* с учетом различных нелинейностей.

Ключевые слова: следящая система, желаемая передаточная функция, пространство состояний, нелинейности переменных состояния, нормальная форма Коши.

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Область научных интересов: теория автоматического управления, цифровая обработка сигналов, цифровая фильтрация, методы аналитического синтеза цифровых следящих систем, математическое моделирование динамических систем.