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On application of algorithms on the basis of least-square and end formula methods in the trajectory measurement processing problems

In this research we obtained an estimate of mean-square deviations in the trajectory measurement processing problem when the estimated trajectory parameters were determined by the end formulas according to the minimum set of parameters measured, and in this case there was no explicit functional relationship between these parameters and the measurement vector. To estimate the mean-square deviations of determining the trajectory according to the end formulas, we proposed to implement the stochastic simulation. We obtained end formulas for the trajectory problems when making angular measurements. The work provides the results of mathematical simulation.

Keywords: least-square method, end formulas, measured and estimated parameters, determining the aircraft coordinates, mathematical simulation.

Problem statement and general solution pattern

The use of the least-square method (LSM) for solving navigation problems in order to determine trajectory parameters of aircraft and other moving objects involves iterative solution to the respective system of normal equations based on measurement results.

Root-mean-square deviations of moving object trajectory parameters are estimated automatically when inverting the matrix of the normal equation system.

Using the measurement-based LSM, the experimental trajectory is determined in accordance with the data of [1–5] by the following formula:

$$\widehat{\mathbf{X}}_{(k+1)} = \widehat{\mathbf{X}}_{(k)} + \Delta\mathbf{X}_{(k+1)};$$

$$\Delta\mathbf{X}_{(k+1)} = (\mathbf{Z}^T_{(k)}\mathbf{W}\mathbf{Z}_{(k)})^{-1}\mathbf{Z}^T_{(k)}\mathbf{W}[\mathbf{Y} - \mathbf{Y}_{\text{pac}}(\widehat{\mathbf{X}}_{(k)})], \quad (1)$$

where $\widehat{\mathbf{X}}_{(k+1)}$, $\widehat{\mathbf{X}}_{(k)}$ – estimates of moving object's experimental trajectory parameters at the $(k+1)$-th and k-th iterations;

$\Delta\mathbf{X}_{(k+1)}$ – correction to moving object's experimental trajectory parameters at the $(k+1)$-th iteration;

$\mathbf{Z}_{(k)}$ – matrix of partial derivatives from the measured parameters as per parameter estimates starting from the k-th iteration;

$\mathbf{Y}$ – measurement vector;

$\mathbf{Y}_{\text{pac}}(\widehat{\mathbf{X}}_{(k)})$ – estimated vector of measured parameters calculated at the k-th iteration;

$\mathbf{W}$ – inverse matrix to the covariance matrix of measurement errors.

The accuracy of the experimental trajectory is determined by the covariance matrix:

$$\mathbf{K} = (\mathbf{Z}^T_{(k)}\mathbf{W}\mathbf{Z}_{(k)})^{-1}, \quad (2)$$

where k corresponds to the number of the last iteration in formula (1).

Assume that n-dimensional random vector $\mathbf{X}$ has a functional relationship with m-dimensional random vector $\mathbf{Y}$:

$$\mathbf{X} = \mathbf{F}(\mathbf{Y}). \quad (3)$$

According to the data of [1]:

$$\mathbf{K}_{\mathbf{X}} \approx \left(\frac{\partial\mathbf{F}}{\partial\mathbf{Y}}\right)\mathbf{K}_{\mathbf{Y}}\left(\frac{\partial\mathbf{F}}{\partial\mathbf{Y}}\right)^T, \quad (4)$$

where $\mathbf{K}_{\mathbf{X}}$, $\mathbf{K}_{\mathbf{Y}}$ – respective covariance matrices;

$\dfrac{\partial\mathbf{F}}{\partial\mathbf{Y}}$ – matrix of partial derivatives from parameters estimated by measured parameters.

Formula (4) is accurate in case of a linear relationship.

Application of the LSM in accordance with books [1–5] is supposed to be a multi-faceted approach to solving problems related to determination of trajectory parameters of n-dimensional vector $\mathbf{X}$ as per measurements of m-dimensional vector $\mathbf{Y}(\mathbf{X})$, as noted above.

For solving the problem of trajectory parameters determination [1] that involves measurements of range, azimuth and elevation angle:

$$x = d\cos\gamma\cos\alpha, \quad y = d\sin\gamma, \quad z = d\cos\gamma\sin\alpha,$$

where x, y, z – aircraft coordinates;

d, α, γ – range, azimuth and elevation angle, respectively, the accuracy of determined parameters can be estimated by formula (4).

Papers [1–3] provide examples of cases when the estimated parameters are determined by end formulas using a minimum set of the measured parameters and at the same time in these cases contain no functional relationship between vector **X** and vector **Y** as per formula (3). In this situation, the accuracy of experimental trajectory cannot be estimated by formula (4). As an example of such navigation problems, we can mention the determination of object coordinates by end formulas, according to three ranges from three beacons [1–3], three differences in ranges from four beacons [2, 3], and direction cosines measured by two phase direction finders [1].

Assume that vector $\hat{\mathbf{X}} = \begin{pmatrix} \hat{x}_1 \\ \ldots \\ \hat{x}_n \end{pmatrix}$ is the experimental trajectory estimated by end formulas, according to m-dimensional vector **Y** (for example, $n = 3$ and $m = 3$), if three trajectory coordinates are to be determined as per three differences in ranges from four beacons in accordance with [2, 3].

To estimate the accuracy (root-mean-square deviations) of experimental trajectory determination, we propose stochastic simulation to be implemented as follows.

Calculation of estimated measurement vector $\mathbf{Y}_{\text{pac}} = \begin{pmatrix} \hat{y}_{1\text{pac}} \\ \ldots \\ \hat{y}_{m\text{pac}} \end{pmatrix}$, corresponding to estimated experimental trajectory **X**.

Simulation of a series of measurements

$$\mathbf{Y}_i = \begin{pmatrix} y_{i1} \\ \ldots \\ y_{im} \end{pmatrix}; \quad y_{ij} = \hat{y}_{im\,\text{pac}} + \varepsilon_{ij}, \qquad (5)$$

where m – dimensionality of measurement vector;

$i = 1, \ldots, N_{ser}$, N_{ser} – amount of series of measurements to be simulated;

ε_{ij} – interference in the i-th test by the j-th coordinate.

Measurements are simulated by adding random interference via a pseudo-random number sensor with the normal distribution law to the estimated values of the measured parameters.

Determination of the corresponding experimental trajectory $\mathbf{X}_i$: for each series of measurements $\mathbf{Y}_i$ using the algorithm of end formulas (statement A)

$$\mathrm{A} : \mathbf{Y}_i \rightarrow \mathbf{X}_i.$$

Based on a series of simulated experimental trajectories, statistic data processing for estimating root-mean-square deviations (RMSD) of trajectory coordinates:

$$\hat{\sigma}_j = \sqrt{\sum_{i=1}^{N_{ser}} \left(x_{ij} - \hat{x}_j\right)^2 \Big/ \left(N_{ser} - 1\right)}, \qquad (6)$$

where $\hat{\sigma}_j$ – estimated RMSD of the j-th coordinate of trajectory ($j = 1, \ldots, n$).

The algorithm of coordinate determination using a minimum set of measurements can be executed based on minimization by applying the method of random search of the sum of squares of deviations of the measured values from the estimated experimental values.

Fig. 1 shows the methods of navigation accuracy estimation using a minimum set of measurements, including end formulas.

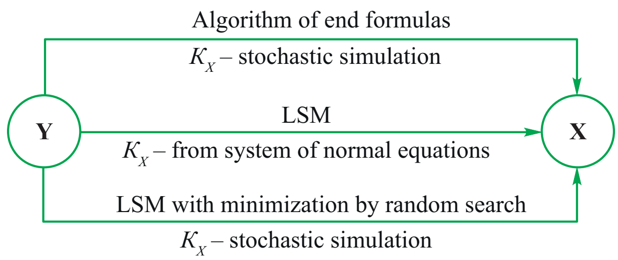

Fig. 1. Method of trajectory determination accuracy calculation using a minimum set of measurements

Determination of kinematic parameters of the trajectory based on angle measurements by means of two measuring facilities

Below we will analyse the problem of object coordinate determination according to azimuths and elevation angles $\alpha_1, \gamma_1, \alpha_2, \gamma_2 \rightarrow x, y, z$ measured by two measuring facilities (MF). This problem is equivalent to the problem of object coordinate determination for two phase direction finders that measure direction cosines in accordance with [1].

Fig. 2 shows the relationship between measured parameters of direction cosines $\cos\theta_x$, $\cos\theta_z$ with azimuth and elevation angle. We can calculate it by the formula:

$$\begin{aligned}\cos\theta_x &= \sin\gamma\cos\alpha,\\ \cos\theta_z &= \sin\gamma\sin\alpha.\end{aligned} \tag{7}$$

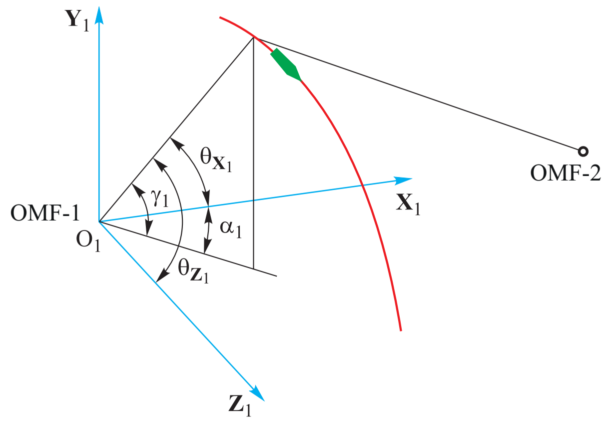

Fig. 2. Transition from azimuth and elevation angle to measurements in the form of direction cosines

Assume that there are measurements of two pairs of direction cosines $\cos\theta_{x_1}$, $\cos\theta_{z_1}$ and $\cos\theta_{x_2}$, $\cos\theta_{z_2}$.

Book [1] contains formulas for determining object coordinates for two phase direction finders which measure direction cosines. A/C coordinates in the local coordinate system (LCS) of the 1st MF can be determined by the formula:

$$x_1 = D_1\cos\theta_{x_1},\quad y_1 = D_1\cos\theta_{y_1},\quad z_1 = D_1\cos\theta_{z_1}. \tag{8}$$

Thus, it is necessary to determine the distance between the A/C and standing point of the 1st MF D_1.

The method of problem solving is illustrated in Fig. 3 [1], showing angles δ, φ, ψ used in subsequent calculations.

Parameter D_1 is determined as follows:

$$D_1 = \frac{b\sin\varphi}{\sin\psi}, \tag{9}$$

where b – distance between two MFs;

$\mathbf{b}_1^0$ – single vector (shown in Fig. 3 and calculated as per [1]):

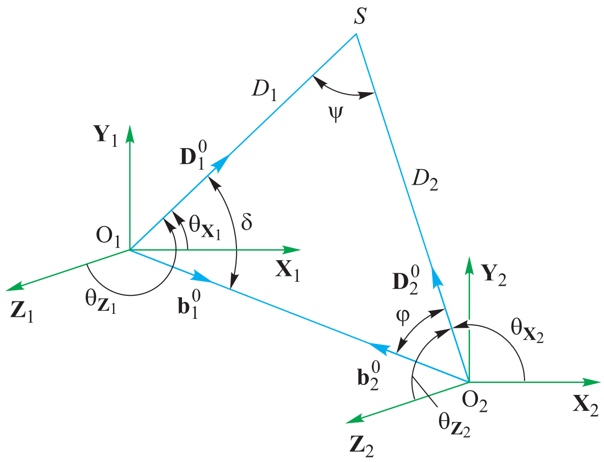

Fig. 3. Determination of object coordinates using two MFs

$$\cos\delta = \mathbf{D}_1^0\boldsymbol{\Phi}_1^T\mathbf{b}_1^0,\quad \cos\varphi = -\mathbf{D}_2^0\boldsymbol{\Phi}_2^T\mathbf{b}_1^0, \tag{10}$$

where $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$ – matrices of direction cosines for transition from the measuring facilities' LCS to the Greenwich coordinate system (GCS);

$$\mathbf{D}_1^0 = \left(\cos\theta_{x_1}, \cos\theta_{y_1}, \cos\theta_{z_1}\right),$$

$$\mathbf{D}_2^0 = \left(\cos\theta_{x_2}, \cos\theta_{y_2}, \cos\theta_{z_2}\right);$$

$$\sin\delta = \sqrt{1-\cos^2\delta},\quad \sin\varphi = \sqrt{1-\cos^2\varphi},$$

$$\sin\psi = \sin\delta\cos\varphi + \cos\delta\sin\varphi. \tag{11}$$

Fig. 4 shows RMSDs during determination of experimental trajectory coordinate x obtained by processing the measurement data with the help of the LSM and end formulas.

According to the data shown in Fig. 4, there is a good agreement between two methods of accuracy (RMSD) estimation based on the LSM and stochastic simulation when solving the problem of trajectory calculation, according to direction cosines measured by two phase direction finders.

It would be interesting to find a mathematical solution to the problem of determination of trajectory parameters according to azimuth and elevation angle from a single MF, plus determination of one angle (azimuth or elevation angle) measured by the second measuring facility.

As noted above, the solution to the problem $\cos\theta_{x_1}, \cos\theta_{z_1}, \cos\theta_{x_2}, \cos\theta_{z_2} \rightarrow x, y, z$ is given in book [1].

The formulas describing the relationship between direction cosines and azimuth and elevation

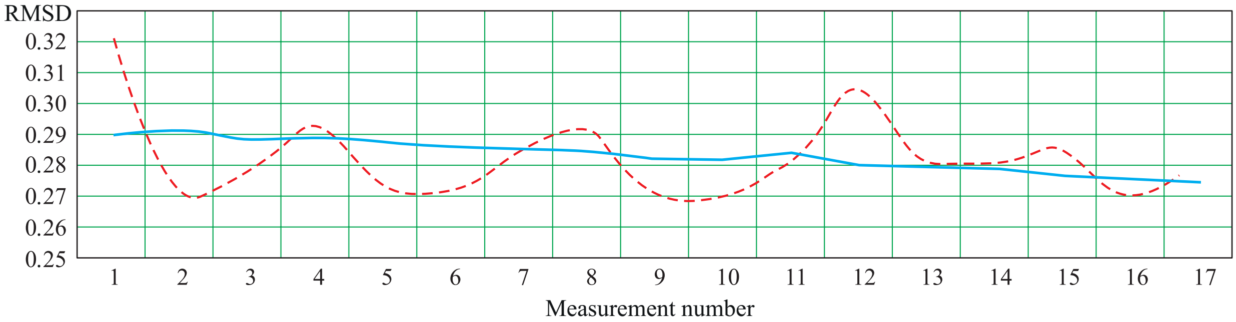

Fig. 4. Coordinate x determination RMSD:
—— – X RMSD (LSM); - - - – X RMSD (end formulas)

angle $\alpha_1, \gamma_1, \alpha_2, \gamma_2 \to \cos\theta_{x_1}, \cos\theta_{z_1}, \cos\theta_{x_2}, \cos\theta_{z_2}$ are represented in the form of equation (7). Therefore, the problem of object coordinate determination according to azimuths and elevation angles $\alpha_1, \gamma_1, \alpha_2, \gamma_2 \to \xi, y, z$ measured by two measuring MFs is solved.

The solutions to both problems are of some interest: $\alpha_1, \gamma_1, \alpha_2 \to x, y, z$ and $\alpha_1, \gamma_1, \gamma_2 \to x, y, z$.

Let us analyse the solutions to the said problems using three methods:

– end formulas;

– least-square method;

– random search algorithm for minimizing the sum of squares.

We get the end formulas for solving the problem $\alpha_1, \gamma_1, \alpha_2 \to x, y, z$ by determining the coordinates of the intersection point of beam l_1, which is determined by angles α_1, γ_1, and plane Π_{α_2}, which is determined by azimuth α_2 and vertical axis O_2Y_2, based on [6–9]. Figs. 5 and 6 illustrate the method of solving both problems, respectively.

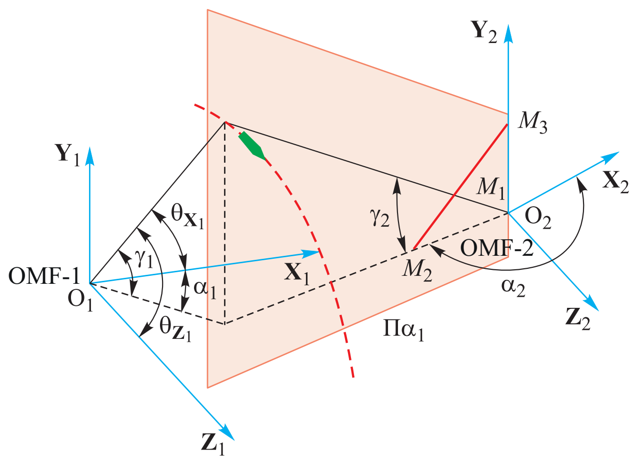

Fig. 5. Method of problem solving $\alpha_1, \gamma_1, \alpha_2 \to x, y, z$

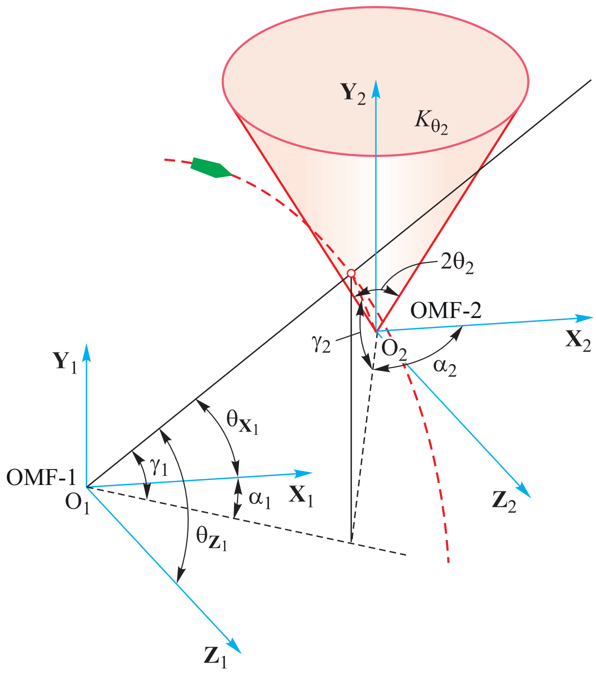

Fig. 6. Method of problem solving $\alpha_1, \gamma_1, \alpha_2 \to x, y, z$

Plane Π_{α_2} can be determined by three points in the LCS of the second MF:

$$M_1 = (x_1, y_1, z_1),\ M_2 = (x_2, y_2, z_2),$$

$$M_3 = (x_3, y_3, z_3),$$

where $x_1 = y_1 = z_1 = 0$;

$$x_2 = \cos\alpha_2,\ \ y_2 = 0,\ \ z_2 = \sin\alpha_2;$$

$$x_3 = z_3 = 0,\ \ y_3 = 1.$$

Plane Π_{α_2} is specified by the following equation:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \qquad (12)$$

In the respective common equation of plane $Ax + By + Cz + D = 0$ for plane Π_{α_2}:

$$A = x_2,\ B = 0,\ C = z_2,\ D = 0.$$

In the LCS of the first MF, beam l_1 lies on a straight line determined by equation in the parametric form:

$$\mathbf{r}_1 = \mathbf{a}_1 t + \mathbf{r}_0, \qquad (13)$$

where $\mathbf{a}_1 = \begin{pmatrix} \cos\theta_{x_1} \\ \cos\theta_{y_1} \\ \cos\theta_{z_1} \end{pmatrix}$; $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

In the LCS of the second MF, the straight line equation has the following form:

$$\mathbf{r}_2 = \mathbf{a}_2 t + \mathbf{r}_{02}, \qquad (14)$$

where $\mathbf{a}_2 = \Phi_2^T \Phi_1 \mathbf{a}_1 = \begin{pmatrix} a_{x_2} \\ a_{y_2} \\ a_{z_2} \end{pmatrix}$; $\mathbf{r}_{02} = \Phi_2^T(\mathbf{b}_1 - \mathbf{b}_2) = \begin{pmatrix} x_{02} \\ y_{02} \\ z_{02} \end{pmatrix}$,

where $\mathbf{b}_1$, $\mathbf{b}_2$ – constant term vectors for transition of measuring facilities from LCS to GCS as per [1].

Intersection of beam l_1 with plane Π_{α_2} corresponds to the parameter:

$$t_1 = \frac{-(Ax_{02} + By_{02} + Cz_{02} + D)}{Aa_{x_2} + Ba_{y_2} + Ca_{z_2}}. \qquad (15)$$

Object coordinates are determined by the formula:

$$\mathbf{r}_2(t_1) = \mathbf{a}_2 t_1 + \mathbf{r}_{02}. \qquad (16)$$

End formulas for solving the problem $\alpha_1, \gamma_1, \gamma_2 \rightarrow x, y, z$ will be obtained by determining the coordinates of the intersection point of beam l_1, which is determined by angles α_1, γ_1 and circular cone K_{θ_2} around axis O_2Y_2 with apex angle $2\theta_2 = 2\left(\frac{\pi}{2} - \gamma_2\right)$.

Cone K_{θ_2} is specified by the equation:

$$\left(x_2^2 + z_2^2\right)\operatorname{ctg}^2\theta_2 = y_2^2. \qquad (17)$$

The condition for intersection of straight line (14) with cone (17) is determined by the equation:

$$\left(\left(a_{x_2} t + x_{02}\right)^2 + \left(a_{z_2} t + z_{02}\right)^2\right)\operatorname{ctg}^2\theta_2 = \left(a_{y_2} t + y_{02}\right)^2. \qquad (18)$$

Further, parameter t is calculated by solving the quadratic equation:

$$At^2 + Bt + C = 0, \qquad (19)$$

where $A = \left(a_{x_2}^2 + a_{z_2}^2\right)\operatorname{ctg}^2\theta_2 - a_{y_2}^2$;

$$B = 2\left(\left(a_{x_2} x_{02} + a_{z_2} z_{02}\right)\operatorname{ctg}^2\theta_2 - a_{y_2} y_{02}\right);$$

$$C = \left(x_{02}^2 + z_{02}^2\right)\operatorname{ctg}^2\theta_2 - y_{02}^2.$$

Object coordinates are calculated by the formula:

$$\mathbf{r}_2\left(t_{1,2}\right) = \mathbf{a}_2 t_{1,2} + \mathbf{r}_{02}, \qquad (20)$$

Here, $t_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ – roots of equation (19).

We exclude the false second solution as per the data of the estimated trajectory.

Therefore, we obtained end formulas for determining the trajectory parameters according to azimuth and cosine measured by one MF and determining one angle (azimuth or elevation angle) measured by the second measuring facility. We should note that these formulas are likely to be represented for the first time.

Trajectory parameters based on azimuths and elevation angles measured by two MFs can be determined using the least-square method for solving the minimization problem:

$$\Phi(x,y,z) = \left(\alpha_1 - \alpha_{1\text{pac}}(x,y,z)\right)^2 + \left(\gamma_1 - \gamma_{1\text{pac}}(x,y,z)\right) + \left(\alpha_2 - \alpha_{2\text{pac}}(x,y,z)\right)^2 + \left(\gamma_2 - \gamma_{2\text{pac}}(x,y,z)\right)^2 \rightarrow \min, \qquad (21)$$

where $\alpha_{i\text{pac}}$, $\gamma_{i\text{pac}}$ – estimated values of MF measured angles;

$i = 1, 2$ – MF numbers;

x, y, z – object coordinates.

Trajectory parameters according to azimuths and elevation angles measured by one MF, as well as one angle (azimuth or elevation angle) measured by the second MF can be determined by solving minimization problem (21), which has no third or fourth term, respectively. There are two methods for solving problem (21). With the first method applied, trajectory parameters are determined iteratively, with the system of normal equations being solved at each step [1–6]. The second method for solving problem (21) involves application of the random search method for minimizing $\Phi(x, y, z)$ [10, 11].

Simulation results are given in the Table. Calculation results are given in the form of deviations Δx, Δy and Δz of experimental trajectory coordinates from their true values for two moments of measurements that correspond to object altitude of 50 and 2 km (numbers of measurements in the Table). The measurement error was taken with RMSD $\sigma_\alpha = \sigma_\gamma = 10$ arcsec. Data processing was performed using end formula (EF) algorithms, via minimization of the sum of squares of measured parameter deviations from their estimated values by applying random search (RS); LSM.

Modelling results

Parameters measured	Method	Measurement number	Deviations, km		
			Δx	Δy	Δz
$(\alpha_1, \gamma_1, \alpha_2, \gamma_2)$	EF	1	–0.013	0.008	–0.013
		2	–0.003	0.010	–0.006
$(\alpha_1, \gamma_1, \alpha_2)$	EF	1	–0.015	0.009	–0.017
		2	–0.004	0.011	–0.007
$(\alpha_1, \gamma_1, \gamma_2)$	EF	1	0.325	–0.133	0.618
		2	–0.002	–0.002	–0.002
$(\alpha_1, \gamma_1, \alpha_2)$	RS	1	–0.015	0.009	–0.017
		2	–0.004	0.012	–0.007
$(\alpha_1, \gamma_1, \gamma_2)$	RS	1	–0.008	0.007	–0.018
		2	–0.014	0.057	–0.032
$(\alpha_1, \gamma_1, \alpha_2, \gamma_2)$	LSM	1	–0.008	0.007	–0.018
		2	–0.003	0.010	–0.007
$(\alpha_1, \gamma_1, \alpha_2)$	LSM	1	–0.015	0.009	–0.017
		2	–0.004	0.011	–0.007
$(\alpha_1, \gamma_1, \gamma_2)$	LSM	1	0.325	–0.133	0.618
		2	–0.002	–0.002	–0.002

The Table data show acceptable matching of estimates for the problems solved by different methods.

Conclusions

1. We specified the method of root-mean-square deviation estimation for determining object coordinates using a minimum set of measured parameters.

2. We obtained end formulas for determining trajectory parameters according to azimuth and elevation angle measured by one measuring facility and for determining one angle (azimuth or elevation angle) measured by the second measuring facility.

3. Performance capability of the proposed algorithms is confirmed through mathematical modelling.

Bibliography

1. *Zhdanyuk B. F.* Osnovy statisticheskoy obrabotki trayektornykh izmereniy. M.: Sovetskoe radio, 1978. 384 s. (Russian)

2. Setevye sputnikovye radionavigatsionnye sistemy / *V. S. Shebshayevich* i dr. M.: Radio i svyaz’, 1993. 408 s. (Russian)

3. *Barabanov O. O., Barabanova L. P.* Matematicheskie zadachi dal’nomernoy navigatsii. M.: FIZMATLIT, 2007. 272 s. (Russian)

4. *Mudrov V. I., Kushko V. L.* Metody obrabotki izmereniy. Kvazipravdopodobnye otsenki. Izd. 2-e. M.: Radio i svyaz’, 1983. 304 s. (Russian)

5. *Tikhonov A. N., Ufimtsev M. V.* Statistical processing of experimental results. M.: Izd-vo Moskovskogo universiteta, 1988. 174 p. (Russian)

6. *Vygodsky M. Ya.* Mathematical handbook. Higher mathematics. M.: Astrel – AST, 2006. 991 p.

7. *Efimov N. V.* Kratkiy kurs analiticheskoy geometrii. M.: FIZMATLIT, 2005. 240 s. (Russian)

8. *Aleksandrov P. S.* Kurs analiticheskoy geometrii i lineynoy algebry. M.: Nauka, 1979. 512 s. (Russian)

9. *Aleksandrov A. D., Netsvetaev N. Yu.* Geometriya: uchebnoe posobie. M.: Nauka. lit., 1990. 672 s. (Russian)

10. *Rastrigin L. A.* Sluchayniy poisk. M.: Znanie, 1979. 64 s. (Russian)

11. Spravochnik po teorii avtomaticheskokgo upravleniya / pod red. *A. A. Krasovskogo.* M.: Nauka, 1987. 712 s. (Russian)

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О применении алгоритмов на основе метода наименьших квадратов и конечных формул в задачах обработки траекторных измерений

Получена оценка среднеквадратических отклонений в задаче обработки траекторных измерений, когда оцениваемые параметры траектории определяются по конечным формулам по минимальному набору измеряемых параметров, при этом в явном виде нет функциональной связи данных параметров с вектором измерений. Для оценки среднеквадратических отклонений определения траектории по конечным формулам предложено осуществлять стохастическое моделирование. Получены конечные формулы для траекторных задач при угловых измерениях. Приведены результаты математического моделирования.

Ключевые слова: метод наименьших квадратов, конечные формулы, измеряемые и оцениваемые параметры, определение координат летательного аппарата, математическое моделирование.

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Область научных интересов: определение параметров движения летательных аппаратов по результатам измерений.